1 Solow: It is all about physical capital accumulation

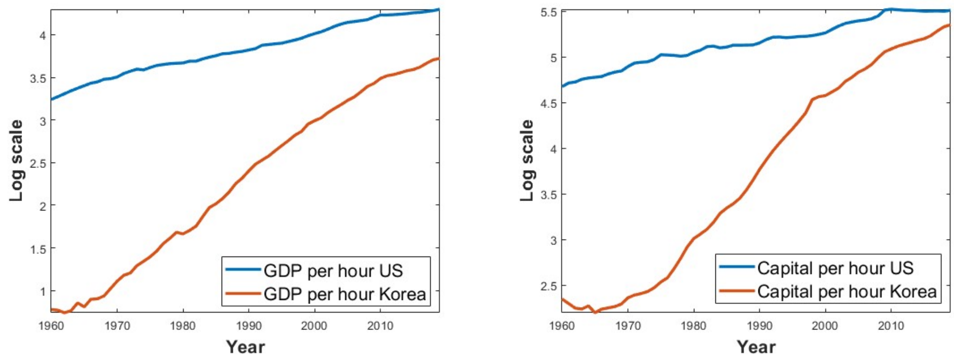
We seen that over the last 150 years, economic growth in advanced economies can be described by exponential growth. What is more, we have seen that there exist peristent differences in output per person across countries. Finally, we have also seen that some countries experience “growth miracles”, i.e., strong positive growth in output per worker over several years. None of these observations are consistent with all economies converging over time to a minimum level of output per worker as predicted by the Malthus model.

Motivated by these (and other) observations, Solow (1956) presents a framework on how to understand the phenomenon of modern economic growth for which he won the Nobel price. As we have discussed in the Malthus model, one key assumption for the model to deliver poverty traps is an exogenous second factor of production. Here, the Solow model makes a key innovation by emphasizing the role of endogenous physical capital accumulation in understanding long run economic phenomena. Introducing physical capital to understand modern economic growth is natural. Today, the equipment, factories, and transportation infrastructure that we use to produce goods and services appear much more important than land relative to the mediveal economy. Moreover, as Figure 1 shows, growth miracles are associated with rapid capital accumulation. That is, the amount of capital that each worker uses in Korea grew much quicker than in the U.S. during the time when output per worker grew much quicker in Korea relative to the U.S. In fact, just visually comparing the mostly agricultural economy of Korea in the 1950s to its industry today that has a large capital-intensive manufacturing sector seems to suggest that capital formation is key for economic development. In this chapter, we will formalize the link between capital accumulation and economic growth and ask what policy implications this has for today’s world.

1.1 Data on modern economic growth

Solow set out to build a model that had a long run stable growth path, a concept that will become clear when we analyze the data. In doing so, he was guided by

Figure 1: Growth in the U.S. and Korea



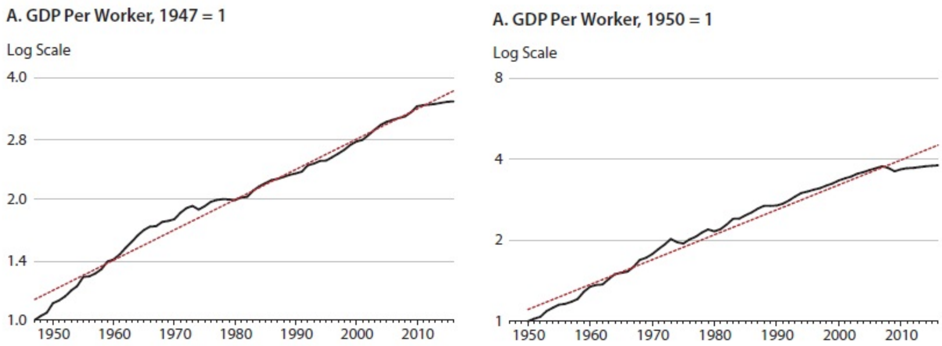
data on income and production that indeed suggested that modern economies were moving along a long run stable trend. These data facts are known as "Kaldor" facts after Kaldor (1961) who was the first to describe these:

1. Output per worker grows at a constant rate over time.
2. Capital per worker grows at a constant rate over time.
3. The capital-to-output ratio is constant over time.
4. Capital has a constant rate of return over time.
5. The share of income going to capital is constant over time.

Kaldor establishes these facts in the 60s for the U.S. Recently, Herrendorf et al. (2019) consider these facts anew and show that these facts still hold in a broad sense. Moreover, they show that these data facts also describe the UK experience. These findings are important for two reasons. First, it gives us additional confidence that these data facts indeed describe a long run stable growth pattern. Second, they are "universally" true, i.e., also hold in other countries than the U.S. Having said that, as we will see below, the data facts are not perfectly stable, in this course, we will mostly treat them as such and consider the Kaldor facts as the benchmark that a model needs to be consistent with.

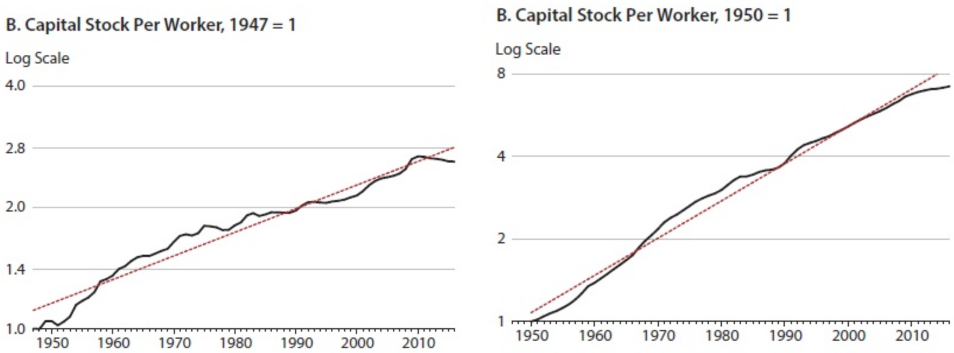
Turning to the data, Figure 2 shows the log of output per worker over time. The left panel displays it for the U.S. and the right panel for the UK. Recall that a linear growth in a log series implies constant exponential growth of the variable.

Figure 2: Constant growth in output per worker



U.S. UK

Figure 3: Constant growth in capital per worker



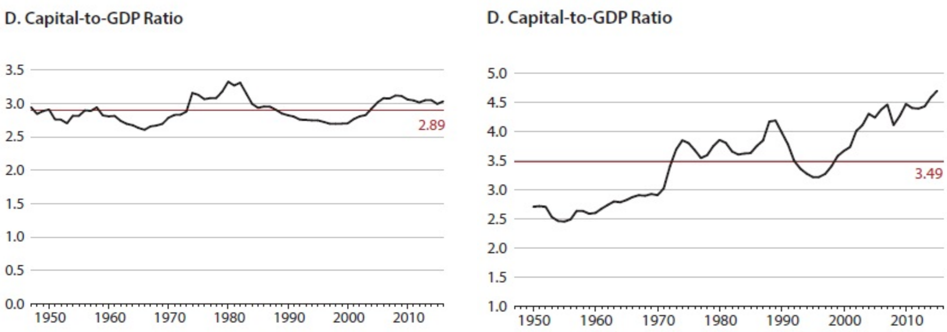
U.S. UK

This is what the figure broadly shows for both economies. To be more precise, the data shows broadly two linear trends, one steeper before 1970 and one flatter after 1970. Again, we will treat the data as if a single trend describes it well, which is still a good approximation. Towards the end of the course, we will ask what factors could explain the growth slow down we observe in 1970.

Figure 3 displays the log of capital per worker over time in the two economies. We see a very similar pattern to the pattern of output per worker: A linear trend is again a good approximation, and we see a small trend break around 1970 since when the growth rate has slowed down somewhat.

With output per worker and capital per worker growing at constant rates, a natural question is which of the two is growing quicker over time. Figure 4 shows

Figure 4: Constant capital-to-output ratio



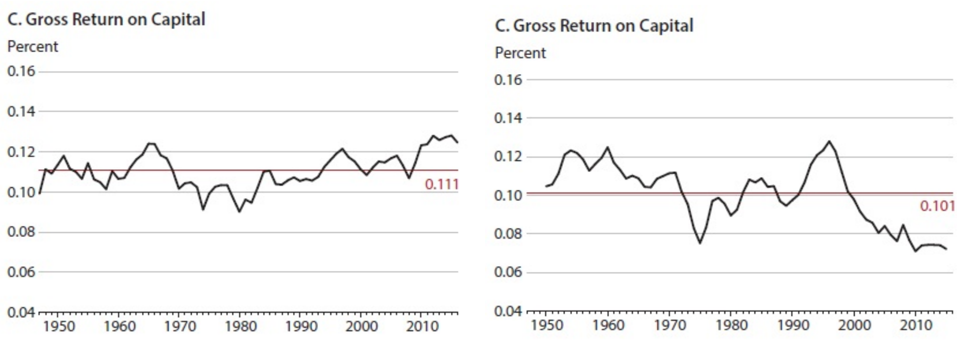
U.S. UK

that the answer is neither: the ratio of capital to output is approximately constant over time. The left panel shows that this is particularly true for the U.S. where the ratio fluctuates around 3.9 since the second World War. The right panel shows that this is not so much true for the UK: The ratio rose from around 2.6 in 1950 to 4.6 by 2015. In this chapter, we will see what kind of policies may contribute to such long run shifts in the capital-to-output ratio and study their implications for output per worker over time.

So far, we only considered data on quantities. Next, we turn to (real) prices. Figure 5 shows that the gross returns to capital have been remarkably stable over time and remarkably similar across countries. Since the second World War, both in the U.S. and the UK, the yearly gross real return on capital has been between 10 and 11 percent yearly. The number may sound surprisingly high given that the real return on government bonds is close to zero since 2010 and has never been as high as 10% since the second World War. Yet, the two observations are not inconsistent as the return on capital measures the return on physical capital (e.g., investing in a company) and not on government bonds. The two differ at least for two reasons. First, to get the net return on capital, we have to subtract from the gross return the depreciation rate of capital which is around 3 percent yearly. Second, investing in physical capital is risky and investors want to be compensated for that risk in form of a risk premium over government bonds.

Finally, we turn to the income share going to capital, that is, the total income going to capital relative to national income. This share is a particularly heated

Figure 5: Constant returns to capital



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topic of discussion since the writing of Karl Marx who formulated his *general law of capitalist accumulation*:

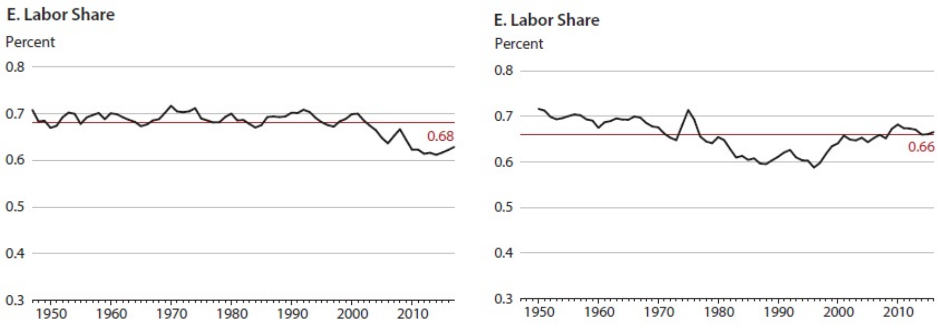
"*It follows therefore that in proportion as capital accumulates, the situation of the worker [...] must grow worse.*" (Marx (1867), p.675)

One way to think about this idea is that a growing capital stock would reduce the share of income going to labor leading to misery among wage earners. This idea was explicitly developed by a later famous Marxist, Karl Kautsky, who wrote in Kautsky (1892):

"*The amount of total capital in capitalist nations is growing faster than the rate of profit is decreasing. The increase in capital is one of the prerequisites for the fall in the rate of profit, and if this falls from 20 [...] to 10 percent, this does not reduce the income of the capitalist whose capital has now increased from one million to [...] four million. His capital income grows from 200,000 to [...] 400,000 marks a year.*"

We have already seen that a fall in the profit rate, despite tremendous capital accumulation, is not consistent with the data since the second World War. One may suspect that this implies a capital share of income that is increasing yet faster (and a labor share that is decreasing yet faster) than predicted by Marxists in the 19th century. However, Figure 6 shows that this is not the case. The labor share of income is close to constant over time at around 67% both in the U.S. and in the UK. In the U.S., it has decreased slightly starting around 2000 but the decline is not big. Below, we will discuss that almost all national income goes either to

Figure 6: Constant capital share in income



U.S. UK

labor or capital in a country like the U.S. Hence, a constant labor share over time also implies that the capital share (1− the labor share) is close to constant over time.

1.2 Modeling a modern economy

A modern economy is substantially more complex than the agrarian economy of the medieval UK. Production usually takes place at the level of firms, instead of the household, and many of these firms are multinational corporations. These firms produce thousands of different goods and services relying on thousands on imports and creating thousands of exports. Moreover, they produce using a large variety of capital goods, ranging from intellectual property rights to factories, and a large variety of labor inputs, ranging from teenagers without work experience or finished education to senior CEOs. Apart from the private sector, the government takes a prominent role in the economy with a share of government spending of national income exceeding 10% in many developed countries. Almost all the goods and services that an economy produces are traded in different markets, instead of being consumed by its producer, and people make decisions about consuming today or in the future, instead of eating whatever is harvested. This description makes already clear that we will need to make several simplifying assumptions to understand aspects of the modern economy.

Let me start with the broadest abstractions and then move to more specific abstractions. First, we will assume a closed economy and leave the discussion of (some form of) trade for latter. Second, we will not model the government and treat it just as part of the private sector. Third, there is only one output good, Y. Fourth, we allow production to take place at the firm level, however, we assume that all factors of production are owned by households. In the case of labor this is obviously true. In the case of capital, it is true that companies like Apple own their factories. However, Apple, and hence also the factories, are ultimately owned by households.

Next, we need to think how firms and households interact in markets (both input and output markets). In Microeconomics, you have seen that a number of market structures exist. To think about what market structure may be appropriate, we consider data from national accounts. Maybe surprisingly, the profit share of national income is relatively small, is is only around 5%. As you have seen in Microeconomics, imperfect competition in any market leads, generally speaking, to positive firm profits while perfect competition implies zero profits. Given the low profit share, we will assume perfect competition in both input and output markets which will simplify our analysis substantially. In particular, we know that the factors of production will earn their marginal products.

Next, let us think about how to model the factors of production. To identify the correct factors of production to focus on, we consider their income shares in national accounts. As we have seen above, labor receives around 68% of national income in developed countries. Piketty and Zucman (2014) show that capital receives another 25% across several developed economies. Put differently, these two factors alone receive almost all of national income which makes it reasonable to abstract from other factors of production. Two observations are in place. First, we will treat land as capital. One may object that different from capital goods such as equipment, the stock of land cannot grow making this an unreasonable simplification. However, fertilizers in agricultural production and high-rise buildings in residential and commercial construction basically allow us to increase the "amount" of usable land. Second, we will not model natural resources. We will explicitly model these later, however, for now, we just make the observation that even in a major resource producing country like the U.S., their share in national income is small. Though national accounts assure us that focusing on only capital and labor is reasonable, there are still thousands of different capital goods and labor inputs that firms use. We will assume that we can just aggregate these to a single capital, K, and a single labor, L, input good. That is, we can just sum the value of all capital input goods and sum the number of workers. We will consider the case of heterogeneous capital goods and different levels of education for labor later.

Firms bring together capital and labor and transforms these into the output good using a specific level of production technology, A. The concept of technological progress is complex. Strictly speaking, it means producing more of the output good using the same amount of input goods. Thinking it about in that way, it is natural to think about improvements in production processes such as:

- better firm organizations, e.g., the assembly line.
- better logistics, e.g., just-in-time delivery.
- better processes, e.g., a faster computer algorithm.

However, this is a too limited way to think about technological progress. Much of technological progress are new products that did not exist in the past. To think about new products as an improvement in technology is not obvious but important. A household today is not six times richer than in 1950 because it owns six washing machines and six dish washers and the household in the 1950s only owned one of each. Instead, it is richer because it owns a cellphone, a good that did not exist in the 1950s. Hence, a good way to think of improvements in A is one of new *ideas*, or better recipes.

Having specified the input factors, we next need to choose a particular form for the aggregate production function, i.e., the function that determines how much output an economy is producing given any amount of labor, capital, and level of technology, $Y = F(K, L, A)$. This is a quite abstract object, however, the assumptions that we have made this far together with the Kaldor facts actually provide a lot of guidance. To be specific, constant income shares imply:

$$\frac{r(t)K(t)}{Y(t)} = \alpha, \tag{1}$$

$$\frac{w(t)L(t)}{Y(t)} = 1 - \alpha. \tag{2}$$

You can see that these equations already contain the left-hand-side of the production function, $Y(t)$, and the inputs $K(t)$ and $L(t)$. However, they also include prices which are obviously not part of the production function. However, given our assumption of competitive markets, we have that these must equal their marginal products and, hence,

$$\frac{\frac{\partial Y(t)}{\partial K(t)} K(t)}{Y(t)} = \alpha, \tag{3}$$

$$\frac{\frac{\partial Y(t)}{\partial L(t)} L(t)}{Y(t)} = 1 - \alpha. \tag{4}$$

These equations look promising as they only contain elements of the production function, its partial derivatives, and constants. In fact, a class of function for which these conditions hold is the Cobb-Douglas production function:

$$Y(t) = K(t)^{\alpha} \left(A(t)L(t)\right)^{1-\alpha}. \tag{5}$$

When you have studied production functions previously, you may have discussed the way that productivity, $A(t)$, enters matters for its interpretation. Here, I have written it as labor-augmenting technology, i.e., $Y(t) = F(K(t), A(t)L(t))$. Yet, note that this is inconsequential in the case of constant returns to scale as we have here. To see this, write

$$Y(t) = K(t)^{\alpha} \left(A(t)L(t)\right)^{1-\alpha} = A(t)^{1-\alpha} K(t)^{\alpha} L(t)^{1-\alpha} = E(t) K(t)^{\alpha} L(t)^{1-\alpha}, \tag{6}$$

with $E(t) = A(t)^{1-\alpha}$ now being Hicks-neutral technology. The way I have written it above will make the math easier. We will use the formulation in Equation (5) going forward because it makes some algebra a little simpler.

One key property of the aggregate production function are diminishing marginal returns to the factor inputs. In words, adding only one of the factor inputs always increases output (the marginal product is always positive), however, the additional output that each additional unit of factor inputs adds is decreasing as we keep adding the factor input. Mathematically, this means that the second partial derivatives of the production function are negative, i.e., the function is concave:

$$\frac{\partial^2 Y(t)}{\partial^2 L(t)} = -\alpha(1-\alpha)K(t)^{\alpha}A(t)^{1-\alpha}L(t)^{-\alpha-1} < 0 \quad (7)$$

$$\frac{\partial^2 Y(t)}{\partial^2 K(t)} = (\alpha-1)\alpha K(t)^{\alpha-2}\left(A(t)L(t)\right)^{1-\alpha} < 0 \quad (8)$$

Diminishing marginal returns are an intuitive outcome once we have aggregated the factors of production to single homogeneous input factors. Take the example of capital in a large warehouse with many workers. Providing workers two instead of only one forklift (forklifts being the single homogenous capital good) will certainly increase output a lot as moving heavy boxes becomes much easier. Also adding the tenth forklift will likely increase output but by less than the second forklift. While the first two forklifts were only moving boxes that were too heavy for workers to move, the tenth forklift will likely start moving boxes that a worker could also have moved (though slower). The 50^{th} forklift might not be used at all but only stand in waiting in case of another one braking down (still a positive marginal product but small).

This intuition, however, is much less obvious once we think about differentiated capital and labor inputs. Consider the case of agricultural production. Certainly, adding tractors to a farm will run into diminishing marginal returns. However, suppose the farmer adds a different capital good, a drone, that monitors crop growth and tells the tractor when and where to harvest. The drone increases the productivity of the tractor (it increases its marginal product), not decreases it! You can think of similar examples with differentiated labor inputs. If you want to build a large dam, adding more and more unskilled workers is again running into diminishing marginal returns to labor. However, adding the first engineer will likely increase the productivity of all the existing unskilled workers instead of decrease it.

The question whether aggregating factors of production was part of the famous Cambridge-Cambridge debate during the 50s. On the one side, you had researchers in Cambridge, UK, who heavily criticized this concept, most prominently in Robinson (1953). On the other side you had researchers in Cambridge, U.S., defending the approach, e.g., Solow (1955). We will follow its defenders and assume that aggregation is possible. However, we should be conscious that this assumption will matter a lot when we think what is behind technological progress. Take the examples from above. We will implicitly interpret adding drones/engineers to existing production technologies as part of technological progress. Later, we will see a model that will model explicitly this type of technological progress through diversification of capital inputs.

Finally, we simplify substantially households' decisions. One of the most important questions in modern macroeconomics is how households trade-off consumption today against tomorrow. This, however, makes the problem quite difficult, Instead, the Solow model abstracts from this complexity and assumes that households save a constant fraction of their income each period. In our model, households receive labor and capital income. Given our assumption of competitive markets, the wage they receive for each unit of labor and the interest rate they receive for each unit of capital are given by their marginal products:

$$w(t) = \frac{\partial Y(t)}{\partial L(t)} = (1-\alpha)K(t)^{\alpha}A(t)^{1-\alpha}L(t)^{-\alpha} \tag{9}$$

$$r(t) = \frac{\partial Y(t)}{\partial K(t)} = \alpha K(t)^{\alpha-1}\left(A(t)L(t)\right)^{1-\alpha}. \tag{10}$$

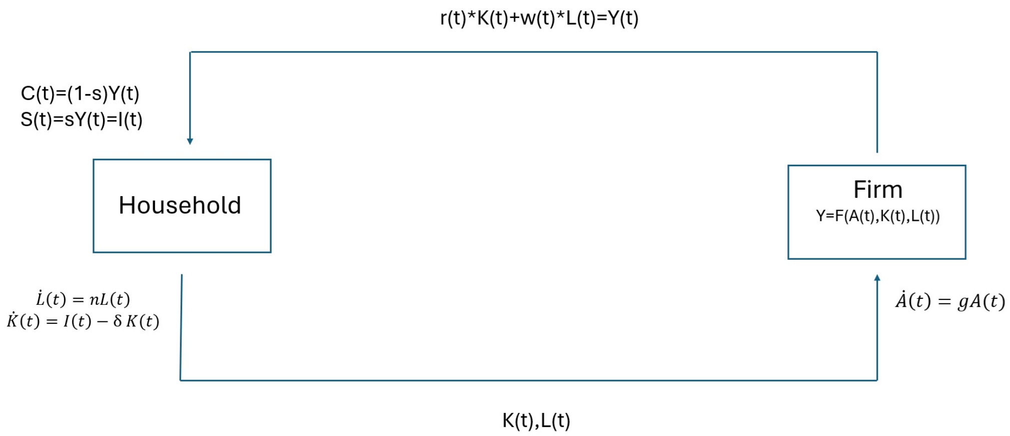
Hence, total household income is

$$r(t)K(t) + w(t)L(t) = Y(t), \tag{11}$$

i.e, households receive the entire production as income from firms. This will be convenient, as we do not need to distinguish between production and household income. That is, aggregate savings are simply $S(t) = sY(t)$.

As we have a closed economy model, new investment must equal aggregate savings: $I(t) = S(t)$. While new investment increases the capital stock, capital depreciation reduces it each period, and the Solow model assumes a constant depreciation rate δ. Therefore, the per-period change in the capital stock is given

Figure 7: Summary of the Solow model



by

$$\dot{K}(t) = S(t) - \delta K(t) \tag{12}$$

$$\dot{K}(t) = sY(t) - \delta K(t) \tag{13}$$

$$\dot{K}(t) = sK(t)^{\alpha} \left(A(t)L(t)\right)^{1-\alpha} - \delta K(t). \tag{14}$$

While the Solow model models endogenous capital dynamics, it simply assumes that the other factors of production follow exogenous growth rates. In particular, we will see later in this chapter that we require an exponential growth rate for technology to be consistent with the Kaldor facts on labor productivity:

$$L(t) = L(0)\exp(nt) \Rightarrow \frac{\dot{L}(t)}{L(t)} = n \tag{15}$$

$$A(t) = A(0)\exp(gt) \Rightarrow \frac{\dot{A}(t)}{A(t)} = g. \tag{16}$$

This completes the description of the Solow model. Figure 7 shows a graphical representation of the economy. Starting with households at the left, they are the owners and accumulators of labor and capital that they rent out to firms on the right side. Firms have a production technology, that changes over time, that converts these input factors into the final output good, $Y(t)$. In the end, all production is paid to households for their factors of production, and households use a fraction of this income to consume and the remaining fraction to accumulate capital.

1.3 The steady state of the economy

We begin again by analyzing the steady state of the economy. We will see later than, as in the Malthus model, the economy converges over time to its steady state. Hence, we can think of the steady state again as the long run outcome of the economy. The steady state analysis will serve two purposes. First, we will see that the economy is indeed consistent with the Kaldor facts which will give us come confidence that the model may be useful to understand modern long run economic growth patterns. Second, we can ask what kind of economic forces can explain long run economic differences in output per worker over time and across economies. That is, we study the model predictions regarding the reasons for developed economies such as the U.S. experiencing exponential growth over time and the reasons for some economies being persistently richer than other economies.

1.3.1 The level of the capital-to-output ration in steady state

As before, we will start our analysis with the steady state of the model. For this, we need to find a variable that has a steady state. We will see bow, in the Solow model, the capital-to-output ratio is such a variable. This feature is obviously not by chance but an outcome of some of the model assumptions that Solow made to be consistent with the fact that this ratio is constant over time in the data. To find the steady state, use the production function to write:

$$z(t) = \frac{K(t)}{Y(t)} = \frac{K(t)}{K(t)^{\alpha} \left(A(t)L(t)\right)^{1-\alpha}} \tag{17}$$

$$= \left(\frac{K(t)}{A(t)L(t)}\right)^{1-\alpha}. \tag{18}$$

A steady state implies that when the variable takes that particular value in period t, it will still have that value in $t + \Delta t$. Hence, we need to derive the dynamics of the capital-to-output ratio over time. To obtain those, we derive its growth rate by using the fact that the derivative of a variable in logs with respect to time is the growth rate of that variable:

$$\ln z(t) = (1-\alpha)\ln K(t) - (1-\alpha)(\ln L(t) + \ln A(t)) \tag{19}$$

$$\Rightarrow \frac{\dot{z}(t)}{z(t)} = (1-\alpha)\frac{\dot{K}(t)}{K(t)} - (1-\alpha)\left(\frac{\dot{L}(t)}{L(t)} + \frac{\dot{A}(t)}{A(t)}\right) \tag{20}$$

$$\frac{\dot{z}(t)}{z(t)} = (1-\alpha)\frac{\dot{K}(t)}{K(t)} - (1-\alpha)\,(n+g)\,. \tag{21}$$

Next, conjecture that a steady state with $\dot{z}(t) = 0$ exists:

$$0 = (1-\alpha)\left(\frac{\dot{K}(t)}{K(t)}\right)^{*} - (1-\alpha)\,(n+g) \tag{22}$$

$$\left(\frac{\dot{K}(t)}{K(t)}\right)^{*} = n+g. \tag{23}$$

If a steady state exists where the capital-to-output ratio is constant, capital needs to grow at the rate of population growth and technological progress, $n+g$.

The next step is to find an expression for the growth rate of the capital stock for Equation (23). To that end, we use the capital accumulation equation:

$$\dot{K}(t) = sK(t)^{\alpha}\,(A(t)L(t))^{1-\alpha} - \delta K(t) \tag{24}$$

$$\frac{\dot{K}(t)}{K(t)} = \frac{s}{z(t)} - \delta. \tag{25}$$

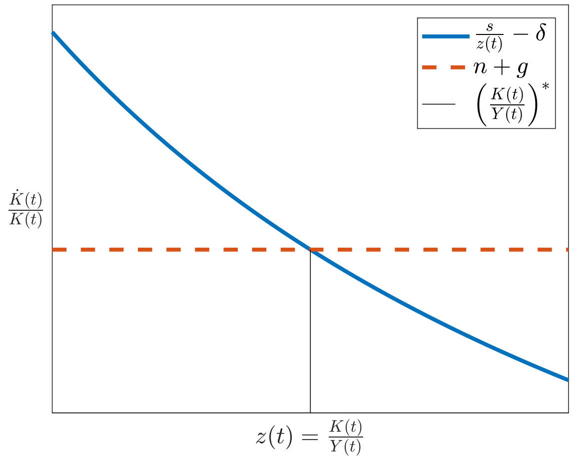
This equation relates the growth rate of the capital stock to the current level of the capital-to-output ratio. Evaluating the eqaution at its steady state $z(t) = z^*$ and combining it with Equation (23) yields our steady state capital-to-output ratio:

$$n+g = \frac{s}{z^*} - \delta \tag{26}$$

$$z^* = \left(\frac{K(t)}{Y(t)}\right)^{*} = \frac{s}{n+g+\delta}. \tag{27}$$

Note, the right-hand-side of the equation are all model parameters that are constant over time. Hence, if the capital-to-output ratio has a steady state (which we have assumed thus far), we have found it.

Figure 8: The steady state



Turning to its economic interpretation, we observe that the steady state capital-to-output ratio is increasing in the savings rate. As we can see in the second steady state condition (25), a higher savings rate implies that the capital stock grows quicker which leads to a larger stock in steady state. Quiet intuitively, the same is true when the capital depreciation rate decreases. We can also see that the steady state capital-to-output ratio is decreasing in the populations growth rate and the growth rate of technology. The first steady state condition (23) makes this clear as an increase of either of the two increases the growth rate of output (the denominator of the capital-to-output ratio).

Figure 8 shows this economic intuition graphically. The graph has the capital-to-output ratio on the x-axis and the growth rate of the capital stock on the y-axis. Steady state condition (23) says that for any capital-to-output ratio, the capital stock needs to grow at $n + g$ to keep the capital-to-output ratio constant which is the horizontal line in the figure. Steady state condition (25) says that the growth rate of the capital stock is falling in a convex fashion in the current level of the capital-to-output ratio which is represented by the downward-sloping line in the figure. The intersection of the two lines gives us our steady state capital-to-output ratio. Note, the two lines can intersect only once and, hence, there exist only one steady state with $\frac{K(t)}{Y(t)} > 0$.

1.3.2 The level of other variables in steady state

Having found a steady state capital-to-output ratio, we can derive the values of other variables in steady state such as output per worker, capital per worker, and consumption per worker. Start with output per worker. To taht end, we rewrite the production function in terms of teh capital-to-output ratio:

$$Y(t) = K(t)^{\alpha} \left(A(t)L(t)\right)^{1-\alpha} \tag{28}$$

$$\frac{Y(t)}{Y(t)^{\alpha}} = \left(\frac{K(t)}{Y(t)}\right)^{\alpha} \left(A(t)L(t)\right)^{1-\alpha} \tag{29}$$

$$Y(t) = \left(\frac{K(t)}{Y(t)}\right)^{\frac{\alpha}{1-\alpha}} A(t)L(t) \tag{30}$$

Which gives output as a function of the capital-to-output ratio. Note, thus far, we have not impossed any steady state condition, i.e., this equation holds in and outside teh steady state. Imposing the steady state condition, we have

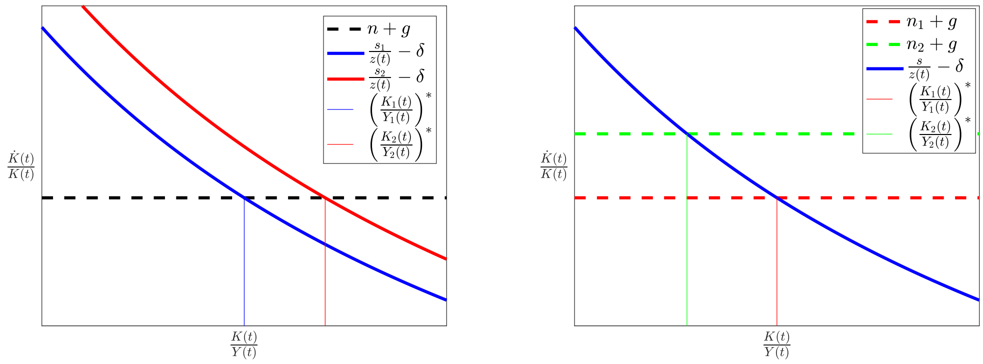
$$Y(t)^{*} = \left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} A(t)L(t) \tag{31}$$

$$y(t)^{*} = \left(\frac{Y(t)}{L(t)}\right)^{*} = \left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} A(t), \tag{32}$$

where, e.g., $Y(t)^{*}$ means that $Y(t)$ itself has no steady state (it still depends on t) but we impose the steady state for the capital-to-output ratio. Put differently, we express output (per worker) when the economy is in its steady state.

Equation (32) highlights that the Solow model, different from the Malthus model, can explain long-run (steady state) differences in output per worker (in the exercises, we will tease out what are the key differences to the Malthus model that deliver us this result). First, and again different from the Malthus model, a higher technology level, $A(t)$, increases output per worker in the long run. Though this is an interesting result, we should keep the limitations of the model in mind. Productivity is just an exogenous process, i.e., the model does not really help us to understand differences across countries. In contrast, the endogenous part that the model helps us to understand are differences in the capital-to-output ratios across countries. According to the Solow model one country is permanently richer

Figure 9: Comparative statics in steady state



than another country because its has a permanently higher capital-to-output ratio. That is, a country is permanently richer because it has a higher savings rate, a lower population growth rate, or a lower capital depreciation rate.

We can use our graphical representation of the steady state to understand how changes in those model parameters affect the steady state capital-to-output ratio and, hence, output per worker. The left panel of Figure 9 displays an increase in the savings rate. Graphically, for any level of the capital-to-output ratio, the growth rate of the capital stock increases. The new steady state is associated with a higher capital-to-output ratio and, hence, a higher output per worker. The right panel displays an increase in the population growth rate. Graphically, for any level of the capital-to-output ratio, the growth rate required to keep the ratio constant, $n+g$, increases. The new steady state is associated with a lower capital-to-output ratio and, hence, a lower output per worker.

Once we have found output (per worker) in steady state, it is straight forward to derive consumption (per worker) and capital (per worker) in steady state. Consumption is simply a constant fraction of output:

$$C(t) = (1-s)Y(t). \tag{33}$$

Hence, consumption per worker in steady state is

$$c(t)^* = \left(\frac{C(t)}{L(t)}\right)^* = (1-s)\left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} A(t). \tag{34}$$

As output per worker, consumption per worker in steady state is increasing in the technology level and the capital-to-output ratio.

To obtain the capital per worker in steady state, we start with the steady state capital-to-output ratio

$$\left(\frac{K(t)}{Y(t)}\right)^* = \frac{s}{n+g+\delta} \tag{35}$$

and plug in the production function:

$$\left(\frac{K(t)}{K(t)^\alpha \left(A(t)L(t)\right)^{1-\alpha}}\right)^* = \frac{s}{n+g+\delta} \tag{36}$$

$$\left(\frac{K(t)^{1-\alpha}}{L(t)^{1-\alpha}}\right)^* = \frac{s}{n+g+\delta} A(t)^{1-\alpha} \tag{37}$$

Hence, we have in steady state:

$$\left(\frac{K(t)}{L(t)}\right)^* = \left(\frac{s}{n+g+\delta}\right)^{\frac{1}{1-\alpha}} A(t). \tag{38}$$

As output per worker, capital per worker in steady state is increasing in the technology level and the capital-to-output ratio.

Finally, we can study factor prices in steady state. The rental price of capital is given by its marginal product:

$$r(t) = \frac{\partial Y(t)}{\partial K(t)} = \alpha K(t)^{\alpha-1} \left(A(t)L(t)\right)^{1-\alpha} = \frac{\alpha}{\left(\frac{K(t)}{Y(t)}\right)} \tag{39}$$

Evaluating the equation in steady state gives us

$$r^* = \frac{\alpha}{\left(\frac{K(t)}{Y(t)}\right)^*} = \alpha \frac{n+g+\delta}{s}. \tag{40}$$

Note, this is again a constant. Hence, the Solow model is consistent with the Kaldor fact of a constant return to capital over time.

Next, consider the wage which is given by

$$w(t) = \frac{\partial Y(t)}{\partial L(t)} = (1-\alpha)A(t)\left(\frac{K(t)}{A(t)L(t)}\right)^{\alpha} \tag{41}$$

$$w(t) = (1-\alpha)A(t)^{1-\alpha}\left(\frac{K(t)}{L(t)}\right)^{\alpha} \tag{42}$$

$$w(t) = (1-\alpha)A(t)\left(\frac{K(t)}{Y(t)}\right)^{\frac{\alpha}{1-\alpha}}. \tag{43}$$

Imposing again a steady state, we have

$$w(t)^{*} = (1-\alpha)A(t)\left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}}. \tag{44}$$

which is not constant because $A(t)$ is not constant. Instead, it is growing at the rate of technology, g, in steady state. Kaldor did not study wages but this fact is also born out by the data.

Once we have factor prices, it is easy to derive factor shares. The shares of income going to capital and labor, respectively, are:

$$\frac{r(t)K(t)}{Y(t)} = \alpha K(t)^{\alpha}\left(A(t)L(t)\right)^{1-\alpha} = \alpha Y(t) \tag{45}$$

$$\frac{w(t)L(t)}{Y(t)} = (1-\alpha)K(t)^{\alpha}\left(A(t)L(t)\right)^{1-\alpha} = (1-\alpha)Y(t). \tag{46}$$

That is, capital obtains α of output and labor obtains $1-\alpha$ of output. Hence, the Solow model is also consistent with the Kaldor facts about constant factor shares. Note, to obtain the constant factor shares, we did not have to impose a steady state. In the Solow model, factor shares are constant inside and outside of the steady state.

1.3.3 Growth in steady state

As explained above, one of the features that motivated Solow in designing the model is the observation that the economy grows over time at a close to constant rate. We begin by describing this growth path in terms of the growth rate of the capital stock. Here, Equations (23) and (25) already tell us that

$$\left(\frac{\dot{K}(t)}{K(t)}\right)^* = n + g \tag{47}$$

$$\frac{\dot{K}(t)}{K(t)} = \frac{s}{z(t)} - \delta. \tag{48}$$

The first equation shows that capital growth at a constant rate, $n + g$, in steady steady state. We can use the second equation to better understand the intuition behind this result. Given our production function, the capital to output ratio is proportional to the marginal product of capital which measures the productivity of capital:

$$\frac{K(t)}{Y(t)} = \frac{K(t)}{K(t)^{\alpha}\left(A(t)L(t)\right)^{1-\alpha}} = \frac{\alpha}{MPK(t)}. \tag{49}$$

That is, a high capital-to-output ratio implies that capital is relatively unproductive. Combining this with Equation (48) gives us

$$\frac{\dot{K}(t)}{K(t)} = \frac{s}{\alpha}MPK(t) - \delta. \tag{50}$$

When the marginal product of capital is high, the additional investment generated from a unit of capital, $\frac{sY(t)}{K(t)} = \frac{s}{\alpha}MPK(t)$ is bigger than the depreciation rate δ, i.e., the capital stock is growing. Technological growth and labor growth do exactly that: they push up the marginal product of capital as Equation (39) shows. The resulting additional capital accumulation decreases again its marginal product, leading to a constant marginal product in steady state:

$$MPK^* = \frac{\alpha}{\left(\frac{K(t)}{Y(t)}\right)^*} = \alpha\frac{n + g + \delta}{s}. \tag{51}$$

Once we know the growth rate of the capital stock, it is straight forward to solve for the growth rate of the capital stock per worker $k(t) = \frac{K(t)}{L(t)}$:

$$\left(\frac{\dot{k}(t)}{k(t)}\right)^* = \frac{\dot{K}(t)}{K(t)} - \frac{\dot{L}(t)}{L(t)} = g. \tag{52}$$

That is, in steady state, capital per capita grows at the rate of technological progress. Hence, the model is consistent with the Kaldor fact of a constant growth rate of capital per capita.

Turning to the growth rate of output (per worker), we have

$$\begin{align} \frac{Y(t)}{Y(t)} &= \frac{\alpha}{1-\alpha}\frac{\dot{z}(t)}{z(t)} + \frac{\dot{A}(t)}{A(t)} + \frac{\dot{L}(t)}{L(t)} \tag{53} \\ \left(\frac{\dot{Y}(t)}{Y(t)}\right)^* &= n + g \tag{54} \\ \left(\frac{\dot{y}(t)}{y(t)}\right)^* &= g. \tag{55} \end{align}$$

Hence, output per capita in steady state also grows at the rate of technological progress. Again, the model is consistent with the Kaldor fact of a cosntant growth rate of output per worker over time. It is again instructive to better understand why output is growing at rate $n + g$. In particular, from the production function we have

$$\begin{align} Y(t) &= K(t)^\alpha \left(L(t)A(t)\right)^{1-\alpha} \tag{56} \\ \frac{\dot{Y}(t)}{Y(t)} &= \alpha\frac{\dot{K}(t)}{K(t)} + (1-\alpha)\left(\frac{\dot{L}(t)}{L(t)} + \frac{\dot{A}(t)}{A(t)}\right). \tag{57} \end{align}$$

The equation may suggest that output should grow at rate $(1-\alpha)(n+g) < n+g$. As we have diminishing marginal returns, an increase in $L(t)$ or $A(t)$ increases output by less than one unit. However, this simple intuition does not take into account what is endogenously happening with $\frac{\dot{K}(t)}{K(t)}$. We have already seen that capital is also growing at $n+g$. Hence, all input factors are growing at rate $n+g$. As we have constant returns to scale, all input factors growing at rate $n+g$ implies that output grows at rate $n+g$.

Finally, for consumption, we have

$$C(t) = (1-s)Y(t) \tag{58}$$

$$\left(\frac{\dot{C}(t)}{C(t)}\right)^{*} = n + g \tag{59}$$

$$\left(\frac{\dot{c}(t)}{c(t)}\right)^{*} = g. \tag{60}$$

Hence, consumption per capita in steady state also grows at the rate of technological progress. A steady state in which all endogenous variables grow at the same rate is referred to as a *balanced growth path*. As discussed above this is exactly what Solow set out to explain: An economy that moves in the long run along a constant growth path.

1.4 The economy outside its steady state

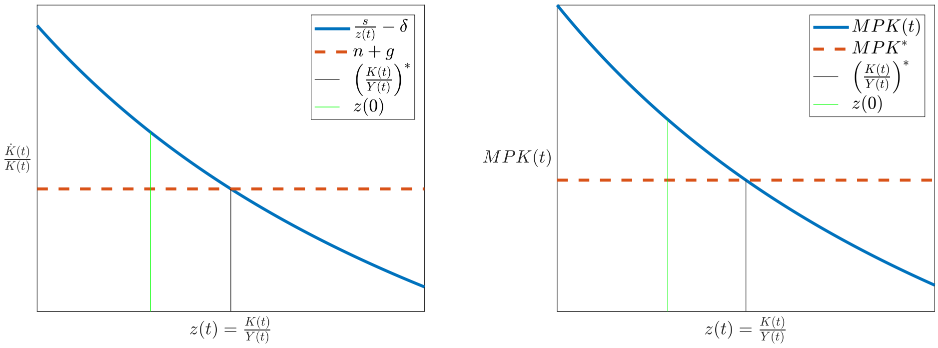
Like in the Malthus model, we would like to understand how the economy behaves outside its steady state. This will serve several purposes. First, thus far, we only assumed that a steady state exists but have not shown that it does. Second, even if it exists, we would like to show that the economy converges to its over time to make it useful as a focal point of long run economic analysis. Third, growth miracles, like the example of Korea above, suggest that not all economies are in their steady state, and that an economy can be outside its steady state for a long time. Hence, we need to understand how economies behave outside their steady state if we want to understand growth miracles.

1.5 Does the economy convergence to its steady state?

We begin by asking whether our economy will converge to its steady state. To that end, we return to the beginning of the chapter but do not impose that the capital-to-output ratio is constant:

$$\frac{\dot{z}(t)}{z(t)} = (1-\alpha)\frac{\dot{K}(t)}{K(t)} - (1-\alpha)(n+g). \tag{61}$$

Figure 10: Convergence to steady state



Now consider an economy with a capital-to-output ratio that is below its steady state level, $z(t) < z^*$. We know already that for such an economy,

$$\frac{\dot{K}(t)}{K(t)} = \frac{s}{z(t)} - \delta > n + g, \tag{62}$$

which can also be seen in the left panel of Figure 10. Hence, Equation (61) tells us that the capital-to-output ratio will be growing over time, i.e., we converge to the steady state over time. Obviously, an analogous argument holds for any economy that starts above its steady state, $z(t) > z^*$. Put differently, our economy converges to its steady state from any starting point $z(0) > 0$. This makes the steady state again a very useful point for policy analysis. That is, the long run behavior of any economy is given by its steady state behavior.

We can think again about the productivity of capital to understand the convergence to steady state. When $z(t) < z^*$, the marginal product of capital is higher than in steady state as shown by the right panel of Figure 10. Hence, as seen before, the additional investment generated from a unit of capital net of depreciation is higher than the growth rate of output in steady state, $n + g$:

$$\frac{s}{\alpha} MPK(t) - \delta > n + g \text{ if } z(t) < z^*, \tag{63}$$

leading to a growth in the capital-to-output ratio. The reverse holds at levels of the capital-to-output ratio above the steady state.

1.6 Solving for the convergence path

Using the dynamics of the capital-to-output ratio, and not imposing that the ratio is constant, and the dynamics of the growth rate of the capital stock, we can solve for the convergence path explicitly:

$$\frac{\dot{z}(t)}{z(t)} = (1-\alpha)\frac{\dot{K}(t)}{K(t)} - (1-\alpha)(n+g) \tag{64}$$

$$\frac{\dot{K}(t)}{K(t)} = \frac{s}{z(t)} - \delta \tag{65}$$

$$\rightarrow \dot{z}(t) = (1-\alpha)s - (1-\alpha)(n+g+\delta)\, z(t). \tag{66}$$

As in the Malthus model, we have a differential equation with a constant. To solve it, define the auxiliary variable $u(t)$ and write $\beta = (1-\alpha)(n+g+\delta)$:

$$u(t) = s(1-\alpha) - \beta z(t) = \dot{z}(t) \qquad \dot{u}(t) = -\beta \dot{z}(t) \tag{67}$$

$$\dot{u}(t) = -\beta u(t) \tag{68}$$

with solution

$$u(t) = u(0)\exp(-\beta t) \tag{69}$$

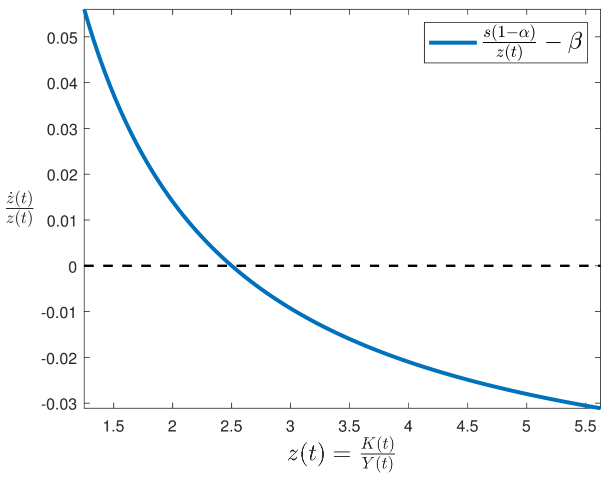
Now substitute again for $u(t)$, and rearrange:

$$s(1-\alpha) - \beta z(t) = [s(1-\alpha) - \beta z(0)]\exp(-\beta t) \tag{70}$$

$$\underbrace{\frac{K(t)}{Y(t)}}_{\frac{\alpha}{MPK(t)}} - \underbrace{\frac{s}{n+g+\delta}}_{\left(\frac{K(t)}{Y(t)}\right)^*} = \left[\frac{K(0)}{Y(0)} - \frac{s}{n+g+\delta}\right]\exp(-\beta t). \tag{71}$$

This equation is key in several aspects. First, it shows that $z(t) = \frac{s}{n+g+\delta}$ is indeed a steady state. To see that, first note that we derived this equation without imposing any steady state assumption. Next, plug in $z(0) = \frac{s}{n+g+\delta}$ and recognize that, hence, $z(t) = \frac{s}{n+g+\delta}$ in all t, i.e., a steady state. Second, the equation shows what we have argued intuitively above: the economy always converges to its steady state. As $t \mapsto \infty$, the right-hand-side of the equation $\mapsto 0$ and, hence, $z(t) \mapsto z^*$.

Figure 11: Growth rate of the capital-to-output ratio



Third, as the equation gives us the value of $z(t)$ in any period t for any initial $z(0)$, we can a solution how the economy converges to its steady state. Notice that the bracket term on the right-hand-side is just a constant and, hence, $z(t) - z^*$ is an exponential growth process. We know that this implies that the process has a constant growth rate of $-\beta$. Put differently, for any $z(0)$, the absolute gap between the capital-to-output ratio and its steady state, $z(t) - z^*$, vanishes at rate β over time.

We can also express the convergence path in terms of the growth rate of the capital-to-output ratio by rewriting the differential equation of $z(t)$:

$$\dot{z}(t) = s(1-\alpha) - \beta z(t) \tag{72}$$

$$\frac{\dot{z}(t)}{z(t)} = \frac{s(1-\alpha)}{z(t)} - \beta. \tag{73}$$

Note, the growth rate is 0 if $z(t) = \frac{s}{n+g+\delta} = z^*$ which should not be a surprise by now. More importantly, the equation is a decreasing, convex function in $z(t)$, and

$$z(t) \mapsto 0 \quad \frac{\dot{z}(t)}{z(t)} \mapsto \infty$$

$$z(t) \mapsto \infty \quad \frac{\dot{z}(t)}{z(t)} \mapsto \beta.$$

Figure 11 displays this curve. It shows that the further the economy is below its steady state (the intersection of the two curves), the more positive is the growth rate of the capital-to-output ratio, and the growth rate becomes very steep when $z(t) - z^*$ becomes very negative. The growth rate becomes more negative the further the economy is above its steady state, and it converges to $-\beta$ as the level of $z(t)$ keeps increasing.

Once we understand these transition dynamics, we use them to understand what happens to output per worker in economies after a policy change. Consider an increase in the savings rate or a decrease in the population growth rate. We have seen that those do not change the growth rate of output per worker in steady state. However, they increase its growth rate temporarily during the transition phase. To see this, note

$$y(t) = z(t)^{\frac{\alpha}{1-\alpha}} A(t) \tag{74}$$

$$\frac{\dot{y}(t)}{y(t)} = \frac{\alpha}{1-\alpha}\frac{\dot{z}(t)}{z(t)} + g. \tag{75}$$

After an increase in the savings rate of a decrease in the population growth rate, $z(0) < z^*$ and, hence, $\frac{\dot{z}(t)}{z(t)} > 0$ and $\frac{\dot{y}(t)}{y(t)} > g$. Qualitatively, we already know that the growth rate will be highest initially and slow down over time. We can also calculate the growth rate of output per worker explicitly by substitution the solution for $z(t)$ into Equation (73) and substituting the result into Equation (75):

$$\frac{\dot{y}(t)}{y(t)} = g + \alpha\left[\frac{s}{\frac{s}{n+g+\delta} + \left[z(0) - \frac{s}{n+g+\delta}\right]\exp(-\beta t)} - (n+g+\delta)\right]. \tag{76}$$

We can write this more compact by dividing the numerator and the denominator by z^*:

$$\frac{\dot{y}(t)}{y(t)} = g + \alpha\left[\frac{n+g+\delta}{1 + \left[\frac{z(0)}{z^*} - 1\right]\exp(-\beta t)} - (n+g+\delta)\right]. \tag{77}$$

We have derived already all the economic intuition for this quantitative equation before but it is worth repeating the key properties. First, when we start in steady state, $z(0) = z^*$, the growth rate of output per worker will equal its steady state growth rate in all periods t: $\frac{\dot{y}(t)}{y(t)} = g$. Second, as time passes, $\exp(-\beta t) \mapsto 0$ and the growth rate converges to its steady state growth rate: $\frac{\dot{y}(t)}{y(t)} \mapsto g$. Third, for any t, the smaller is $\frac{z(0)}{z^*}$, i.e., the further the economy is below its steady state value, the higher is the growth rate of output per worker.

Figure 12: Output per worker dynamics

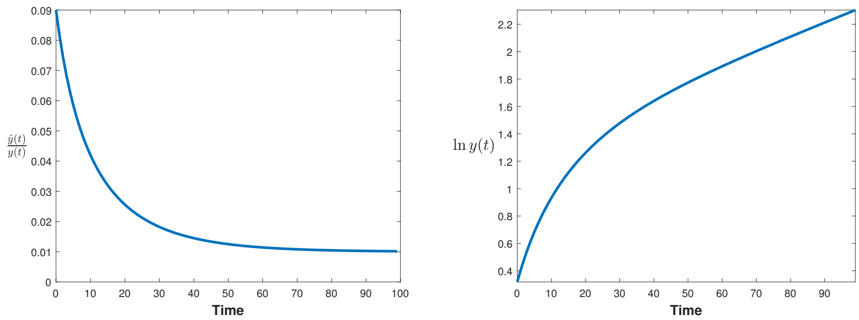


Figure 12 displays the dynamics of output per worker over time for an economy that starts below its steady state. The left panel shows that the growth rate in output per worker is initially high and converges over time to its steady state growth rate g. The right panel shows the implications for the log of output per worker. The high initial growth rate that slows down over time implies that log output per worker is a concave function initially and converges over time to a linear function (with slope g). Importantly, this behavior is qualitatively not dissimilar to the one we observe for Korea in Figure 1 after 1982. However, these transition dynamics cannot explain why the growth rate of output per worker is high and close to linear from 1965 to 1982 in Korea.

It is important to reflect on the ultimate source for how the model explains growth miracles like Korea. The model rationalizes these to result exclusively from rapid capital accumulation, i.e., an increase in the capital-to-output ratio. Along the entire transition path, the growth rate of technology is constant, g. Later, we will see models that also allow for transition dynamics in the growth rate of technology.

1.7 Introducing human capital

So far, we assume that labor is an input that is comparable across time and countries. This may be a poor assumption. We know from a large literature in labor economics that wages of a worker, a proxy for her/his productivity, varies substantially with the education level of workers. Importantly, educations levels vary substantially over time and across countries. While in the generation of our grand parents most people finished at most secondary education, today's generation has a large fraction of people going to university. Even more stark, even today, the average years of schooling are only around 4 years in the poorest countries in the world. Hence, it may be promising to introduce education into our model to understand the long run behavior of output per worker within a country, output differences across countries, and transition dynamics after educational reforms.

To introduce human capital, we make a small change to the production function:

$$Y(t) = K(t)^{\alpha} \left(A(t)H(t)\right)^{1-\alpha} \tag{78}$$

$$H(t) = \exp(\psi\, u)L(t), \tag{79}$$

where $L(t)$ is the amount of labor, and $H(t)$ is the amount of total human capital. Total human capital not only depends on the amount of labor but also in the time invested in education, u. ψ is the quality of schooling. To understand its interpretation, note that

$$\frac{\partial \ln H(t)}{\partial u} = \psi. \tag{80}$$

That is, a change in u translates into ψ percent more human capital. There is a large body of micro-econometric analysis that estimates this semi-elasticity. Harmon et al. (2000) review the literature and conclude that one additional year of education provides about a 9% wage increase. Obviously, ψ, i.e., the quality of the education sector, is neither constant in time nor constant across countries. For example, today, the average class size is substantially smaller than classes when your grand parents went to school. Similarly, class sizes are often much bigger and teachers worse educated in developing compared to developed countries.

1.7.1 The steady state

As always, we begin by analyzing the steady state of the model. We will proceed in exactly the same steps as before. First, to find the first steady state condition of the capital-to-output ratio, we use the production function to write:

$$z(t) = \frac{K(t)}{Y(t)} = \frac{K(t)}{K(t)^{\alpha}\left(A(t)H(t)\right)^{1-\alpha}} \tag{81}$$

$$= \left(\frac{K(t)}{A(t)H(t)}\right)^{1-\alpha}. \tag{82}$$

Next, as before, we derive the growth rate of the capital-to-output ratio:

$$\ln z(t) = (1-\alpha)\ln K(t) - (1-\alpha)(\ln H(t) + \ln A(t)) \tag{83}$$

$$\Rightarrow \frac{\dot{z}(t)}{z(t)} = (1-\alpha)\frac{\dot{K}(t)}{K(t)} - (1-\alpha)\left(\frac{\dot{H}(t)}{H(t)} + \frac{\dot{A}(t)}{A(t)}\right) \tag{84}$$

$$\frac{\dot{z}(t)}{z(t)} = (1-\alpha)\frac{\dot{K}(t)}{K(t)} - (1-\alpha)\left(n+g\right). \tag{85}$$

because ψ and u are constant and, hence, $\frac{\dot{H}(t)}{H(t)} = \frac{\dot{L}(t)}{L(t)}$. Finally, if a steady state for the capital-to-output ratio exists, the growth rate of the capital stock in steady state must be

$$\left(\frac{\dot{K}(t)}{K(t)}\right)^{*} = n + g, \tag{86}$$

which is the same condition as in the model without human capital.

To derive the second steady state condition, we use again the capital accumulation equation:

$$\dot{K}(t) = sK(t)^{\alpha}\left(A(t)H(t)\right)^{1-\alpha} - \delta K(t) \tag{87}$$

$$\frac{\dot{K}(t)}{K(t)} = \frac{s}{z(t)} - \delta. \tag{88}$$

Hence, if a steady state exists, we must have, as in the model without education,

$$n + g = \frac{s}{z^*} - \delta \tag{89}$$

$$z^* = \frac{s}{n + g + \delta}. \tag{90}$$

We can do an analogous steady state analysis to the model without human capital. Here, I will highlight only two insights. First, let us use again the production function to solve for output per worker in steady state:

$$Y(t) = K(t)^\alpha \left(A(t)H(t)\right)^{1-\alpha} \tag{91}$$

$$\frac{Y(t)}{Y(t)^\alpha} = \left(\frac{K(t)}{Y(t)}\right)^\alpha \left(A(t)H(t)\right)^{1-\alpha} \tag{92}$$

$$Y(t) = \left(\frac{K(t)}{Y(t)}\right)^{\frac{\alpha}{1-\alpha}} A(t)H(t) \tag{93}$$

$$Y(t)^* = \left(\frac{s}{n + g + \delta}\right)^{\frac{\alpha}{1-\alpha}} A(t)H(t) \tag{94}$$

$$y(t)^* = \left(\frac{Y(t)}{L(t)}\right)^* = \left(\frac{s}{n + g + \delta}\right)^{\frac{\alpha}{1-\alpha}} A(t) \exp(\psi\, u) \tag{95}$$

Observe that output per worker is increasing in the amount and quality of education. A more educated workforce is more productive and, thereby, allows each worker to produce more. By naively looking at the production function, one may conjecture that the effect of education on output (per worker) would be $\exp(\psi\, u)^{1-\alpha} < \exp(\psi\, u)$. This logic misses that also the capital stock of the economy is affected by education. To see this, start with the steady state capital-to-output ratio

$$\left(\frac{K(t)}{Y(t)}\right)^* = \frac{s}{n + g + \delta} \tag{96}$$

and plug in the production function:

$$\left(\frac{K(t)}{K(t)^{\alpha}\left(A(t)H(t)\right)^{1-\alpha}}\right)^{*}=\frac{s}{n+g+\delta} \tag{97}$$

$$K(t)^{*}=\frac{s}{n+g+\delta}A(t)L(t)\exp(\psi\, u). \tag{98}$$

Ceteris paribus, a more educated workforce increases the marginal product of capital:

$$MPK(t)=\alpha K(t)^{\alpha-1}\left(A(t)H(t)\right)^{1-\alpha} \tag{99}$$

$$=\frac{\alpha}{\frac{K(t)}{Y(t)}}. \tag{100}$$

The higher marginal product of capital leads to additional capital accumulation until the marginal product has reached again its steady state level $\frac{\alpha}{z^{*}}$. Taken together, as both $H(t)$ and $K(t)$ increase in steady state one-to-one with $\exp(\psi\, u)$, $Y(t)$ also increases in steady state one-to-one with $\exp(\psi\, u)$.

Finally, as before, we can ask again how the economy growth over time in steady state. As education is assumed to be constant, nothing really changes. In fact, we have already seen that:

$$\left(\frac{\dot{K}(t)}{K(t)}\right)^{*}=n+g \tag{101}$$

$$\left(\frac{\dot{k}(t)}{k(t)}\right)^{*}=g. \tag{102}$$

That is, capital per capita (and output/consumption per capita) grows at the rate of technological progress.

1.7.2 Transition dynamics

We have seen above that changes in the population growth rate and saving rate have rich transition dynamics for output per worker. Now, we are in a position to analyze transition dynamics resulting from changes in education. Output per worker is:

$$y(t) = \frac{Y(t)}{L(t)} = \left(\frac{K(t)}{Y(t)}\right)^{\frac{\alpha}{1-\alpha}} A(t)\exp(\psi\, u). \tag{103}$$

As the education variables are constant, the growth rate of output per worker is again

$$\frac{\dot{y}(t)}{y(t)} = \frac{\alpha}{1-\alpha}\frac{\dot{z}(t)}{z(t)} + g. \tag{104}$$

As we have seen before, the dynamic equation for the capital-to-output ratio is the same. Therefore, its solution is also the same:

$$z(t) = \frac{s}{n+g+\delta} + \left[z(0) - \frac{s}{n+g+\delta}\right]\exp(-\beta t), \tag{105}$$

which is again independent of education variables. Hence, one may be tempted to think that changes in education variables imply no transition dynamics, i.e., the economy jumps directly to its new steady state. This conclusion is wrong. Increasing the time spend in education, u, or the quality of education, ψ, in a period $t = 0$ increases output in period 0:

$$Y(0) = K(0)^{\alpha}\left(A(0)L(0)\exp(\psi\, u)\right)^{1-\alpha}. \tag{106}$$

Hence,in period 0, the capital-to-output ratio falls, and $z(0) < z^* = \frac{s}{n+g+\delta}$. As a result, the capital-to-output ratio will grow over time back to its steady state implying that output per worker grows quicker than in steady state.

It is worth to think again about the economic intuition for the higher than steady state growth rate in output per worker. As discussed above, an increase in education increases the marginal product of capital above its steady state level. As a result, the net return per unit of investment per capital, $\frac{s}{\alpha}MPK(t) - \delta$, is higher than the growth rate of output in steady state, $(n+g)$, leading to an increase in the capital to output ratio.

1.8 The "optimal savings rate"

The Solow model takes the savings rate as exogenous. Nevertheless, we can ask what savings rate would be desirable from a social point of view. This forces us to think about a welfare objective of a society. The welfare objective of a country is by no way obvious to define, and standard Microeconomics tells us that aggregating individual preferences is not always possible. One may be tempted to think that maximizing output per worker would be a sensible welfare objective. However, this would imply setting $s = 1$, i.e., maximizing the capital stock which implies zero consumption, a world few people would want to live in. Instead, in economics, we usually believe that individual well-being increases in the consumption level. Hence, a useful welfare measure may be the steady state, i.e., the long-run level of consumption per worker:

$$\left(\frac{C(t)}{L(t)}\right)^* = (1-s)\left(\frac{Y(t)}{L(t)}\right)^* = (1-s)\left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} A(t)\exp(\psi\, u) \qquad (107)$$

The savings rate maximizing steady state consumption per worker is referred to as the golden rule s_{Gold}. Taking the first-order condition yields $s_{Gold} = \alpha$. Intuitively, the more important is capital in the production process, the more we ought to save. We can use the expression of the marginal product of capital in steady state to derive further economic intuition for the result: Recall that in steady state, the marginal product of capital is

$$MPK^* = \alpha\frac{n+g+\delta}{s}, \qquad (108)$$

which is decreasing in s, i.e., a higher savings rate leads to more capital in steady state and, hence, a lower marginal product. Substituting $s_{Gold} = \alpha$, we obtain

$$MPK^* - \delta = n + g. \qquad (109)$$

The left-hand side is the net marginal gain of an additional unit of capital, i.e., the amount of additional output after capital depreciation that is available for consumption and investment. The right-hand-side is the implied marginal cost of operating the economy with one additional unit of capital, i.e., the additional savings required to keep the steady state capital-to-output ratio constant. For example, raising the savings rate above α implies that the additional net output that this unit of capital provides is smaller than the additional savings required to maintain the higher capital stock in steady state and, thus, consumption per worker declines in steady state.

1.8.1 Is Spain saving enough?

We can use national account data to asses whether Spain is saving more or less than s_{Gold}. Spain has a capital-to-output ratio of around $\frac{K(t)}{Y(t)} = 2.75$. Moreover, firms invest about 10 percent of yearly output which, in steady state means $0.1Y(t) = \delta K(t)$. Combing these two facts, implies that the yearly depreciation rate is 3.6%. Moreover, we have that the capital share of income is $\frac{r(t)K(t)}{Y(t)} = 30\%$ which, according to our model implies $\frac{MPK(t)K(t)}{Y(t)} = 30\%$. Combing this equation with a capital-to-output ratio of 2.75 implies that $MPK = 0.11$. Finally, output growth in Spain is about 3 percent annually which according to our model is equal $n + g$. Hence, we have $MPK - \delta = 0.074 > n + g = 0.03$. As the net returns on capital are too high for the golden rule, this implies Spain is saving too little. This finding is not unique to Spain but holds for many economies in the developed world. One can draw two conclusions from this finding. Either, many economies are saving too little and governments should provide incentives to increase savings. Alternatively, one can question the premise that the golden rule is optimal for a society. The most obvious objection is that it ignores time discounting. A wealth of micro data suggest that people are impatient and prefer consumption today over consumption in the future. If that is the case, maximizing long-run consumption per worker will not be optimal. Society would prefer to increase consumption somewhat today at the cost of decreasing it somewhat in the long run steady state. When solving a dynamic version of the Solow model, where the households discount the future, one obtains that one can rationalize the high MPK when the time discount factor is about 4%, i.e., approximately the gap between the net returns on capital and the output growth rate. Note, it is this difference between the net returns on capital and the growth rate of an economy that Piketty (2015) is seeing as a key driver for increasing inequality between households over time.

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